(g) $0x = 0$

(h) $1x = x$

We define subtraction as

$$x - y = x + (-1)y,$$

then $x - x = 0.$ We can also write $-x$ for (–1) *x* and *x/a* for (1/*a*)*x*.

The Dot Product: Dot product of two vectors is defined as

$$x.y = x_1y_1 + x_2y_2 + ... + x_ny_n, \qquad (3)$$

where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n).$ Observe that the dot product (3) is a scalar. For that reason *x.y* is sometimes called the scalar product of *x* and *y*. It is also sometimes called the inner product of *x* and *y*. The dot product satisfies certain conditions that we summarize in next theorem.

Theorem 2 : If $x, y, z \in$ R^n and $a \in$ R, then

(a) $x.y = y.y$ (commutative law)

(b) $x.(y + z) = x.y + x.z$ (distributive law)

(c) $(ax).y = a(x.y)$

Observe that $0.x = 0,$ but that *x.y* can equal 0 without either *x* and *y* being 0 [E.g., in R^2, (1, 1). (1, –1) = 1 –1 = 0].

Two nonzero vectors *x* and *y* for which *x.y* = 0 are said to be orthogonal vectors. Geometrically, two orthogonal vectors in R^n are perpendicular.



CHECK YOUR PROGRESS 1

Q 1: For *x* = (1, 2, 3) and *y* = (2, 3, 4), find *x.y*

Q 2: Prove that *x.x* = 0 if and only if *x* = 0

Q 3: For f, g in $C[0, 2\pi]$, let $f.g = \int_0^{2\pi} f(t)g(t)dt.$

(a) Show that "." satisfies the conditions of Theorem 2.

10.4 EXPLICIT AND IMPLICIT FUNCTIONS

If we consider a set of *n* independent variables *x, y, z*, ..., *t* and one dependent variable u, the equation

$$u = f(x, y, z, ..., t),$$

denotes the functional relation. In this case if $x_1, y_1, z_1, ..., t_1$ arbitrarily assigned values of the independent variables, the corresponding values of the dependent variable *u* are determined by theh function relation.

The function represented by equation (4) is an explicit function but where several variables are concerned it is rarely possible to obtain an equation expressing one of the variables explicitly in terms of the others. Thus most of the functions of more than one variable are implicit functions, that is to say we are given a functional relation.

$$\phi(x, y, z, ..., t) = 0 \qquad (5)$$

connecting the *n* variables *x, y, z, ..., t* and is not in general possible to solve this equation to find an explicit function which expresses one of these variables say *x*, in terms of the others *n* – 1 variables.

In this unit we shall be mainly concerned with the explicit functions.

An Explicit Function of Two Variables : If *x, y* are two independent variables and a variable *z* depends for its values on the values of *x, y* by a functional relation

$$z = f(x, y) \qquad (6)$$

then we say *z* is a function *x, y*. The ordered pair of numbers *x, y* is called a point and the aggregate of the pairs of numbers (*x, y*) is said to be the domain (or region) of definition of the function.

When the domain of definition is bounded by a closed curve *C*, it is said to be closed if *f* is defined for all points within and on the curve *C*; but open or unclosed when the function is defined for points within but not on the curve *C*.

The neighbourhood of a point : The set of values x_1, y_1 other than *a, b* that satisfy the conditions

$$|x_1 - a| \le \delta, |y_1 - b| \le \delta$$

where δ is an arbitrarily small positive number, is said to form a neighbourhood of the point (*a, b*).

Thus a neighbourhood is the square

$$(a - \delta, a + \delta; b - \delta, b + \delta)$$

where *x* takes any value from $a - \delta$ to $a + \delta$ except *a*, and *y* from $b - \delta$ to $b + \delta$ except *b*.

This is not the only way of specifying a neighbourhoodo of a point. There can be many other, though equivalent ways; for example the points inside the circle $x^2 + y^2 = \delta^2$ may be taken as a neighbourhood of the point (0, 0).

10.5 LIMIT AND CONTINUITY OF FUNCTION FROM R^N TO R

Definition : The limit of the function $f : D \subset R^n \to R$, with $n \in N$, at the point $\hat{P} \in R^n$ is the number $L \in R$, denoted as $\lim_{P \to \hat{P}} (P) = L$ if and only if for every $\in > 0$ there exists $\delta > 0$ such that

$$0 < \left| P - \hat{P} \right| < \delta \Rightarrow \left| f(P) - L \right| < \in \qquad (7)$$

Remarks :

(a) In Cartesian coordinates $P = (x_1, x_2, ..., x_n),\ \hat{P} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$. Then $| P - \hat{P} |$ is the distance between points in R^n,

$$| P - \hat{P} | = | \overrightarrow{PP} | = \sqrt{(x_1 - \hat{x}_1)^2 + (x_2 - \hat{x}_2)^2 + ... + (x_n - \hat{x}_n)^2}$$

(b) $| f(P) - L |$ is the absolute value of real numbers.

The limit of functions *f :* $R^2 \to R$ **:** A function *f* is said to tend to a limit *L* as a point (*x, y*) tends to the point (x_0, y_0) if for every $\in > 0$ there exists $\delta > 0$ such that

$$0 < | x - x_1 | < \delta,\ 0 < | y - y_0 | < \delta \Rightarrow | f(x, y) - L | < \in \qquad (8)$$

equivalently,

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow | f(x, y) - L | < \in \qquad (9)$$

In other words, a function *f* tends to a limit *L*, when tends to the point (x_0, y_0) if for every positive number $\in$, there corresponds a neighbourhood *N* of (x_0, y_0) such that

$$\left| f(x, y) - L \right| < \in,$$

for every point (*x, y*) other than (x_0, y_0) of the neighbourood *N*.

Symbolically we then write

$$\lim_{(x,y) \to (x_0, y_0)} f(x, y) = L, \qquad (10)$$

L is the limit (the double limit or the simultaneous limit) of *f* when *x, y* tend to *a* & *b* simultaneously.

Computing a limit by the definition:

Example 1 : Use the definition of limit to compute $\lim_{(x,y)\to 0,0)} \frac{2yx^2}{x^2+y^2}$

Solution: The function $f(x, y) = \frac{2yx^2}{x^2+y^2}$ is not defined at (0, 0).

First : Guess what the limit L is.

Along the line $x = 0$ the function is $f(0,y) = \frac{0}{y^2} = 0.$

Therefore, if L exists, it must be $L = 0$. Given $\in$, find δ.
Fix any number $\in > 0.$ Given that $\in$, find a number $\delta > 0$ such that

$$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \Rightarrow \left| \frac{2yx^2}{x^2+y^2} \right| < \in$$

i.e., $0 < \sqrt{x^2+y^2} < \delta \Rightarrow \left| \frac{2yx^2}{x^2+y^2} \right| < \in$

Recall : $x^2 \le x^2 + y^2,$ that is, $\frac{x^2}{x^2+y^2} \le 1.$ Then

$$\left| \frac{2yx^2}{x^2+y^2} \right| = (2 \mid y) \mid \frac{x^2}{x^2+y^2} \le (2 \mid y \mid) = 2\sqrt{y^2} \le 2\sqrt{x^2+y^2}$$

Choose $\delta = \frac{\in}{2} \cdot \sqrt{x^2+y^2} \le \delta,$ then $\left| \frac{2yx^2}{x^2+y^2} \right| < 2\delta = \in$

We conclude that $L = 0$.

Example 2 : Show that

$$\lim_{(x,y)\to 0,0)} xy \frac{x^2-y^2}{x^2+y^2} = 0$$

Solution: Taking $x = r\cos\theta$ and $y = r\sin\theta$

$$\left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| = \left| xy\frac{x^2-y^2}{x^2+y^2} \right| = \left| r^2 \sin\theta\cos\theta\cos 2\theta \right|$$

$$= \left| \frac{r^2}{4}\sin 4\theta \right| \le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \in$$

if

$$\frac{x^2}{4} < \frac{\in}{2}, \frac{y^2}{4} < \frac{\in}{2}$$

or if $|x| < \sqrt{2\in} = \delta, |y| < \sqrt{2\in} = \delta$

Thus for $\in > 0, \exists \delta > 0$ such that

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \in, \text{ when } |x| < \delta, |y| < \delta$$

$$\Rightarrow \lim_{(x,y)\to 0,0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

10. Algebra of Limits

In this section, we look at the algebra of limits for the four arithmetic operations viz. addition, subtraction, multiplication and division.

Theorem 3 : If *f, g* be two functions defined on some neighbourhood of a point (x_0, y_0) such that $\lim f(x,y) = L$ and $\lim g(x,y) = M,$ when $(x,y) \to (x_0, y_0),$ then

(a) $\lim(f+g) = \lim f + \lim g = L + M$

(b) $\lim(f-g) = \lim f - \lim g = L - M$

(c) $\lim(f.g) = \lim f . \lim g = L.M$

(d) $\lim\frac{f}{g} = \frac{\lim f}{\lim g} = \frac{L}{M},$ provided $M \neq 0,$ when $(x,y) \to (x_0, y_0).$

The proofs are exactly similar to those of the corresponding theorems for a single variable.

Remarks: The definition implies that there must be no assumption of any relation between the independent variables as they tend to their respective limits. For instance take

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

and find the limit when $(x,y) \to (0,0).$

If we put $y = m_1 x$ and let $x \to 0,$ we get the limit to be equal to $\frac{m_1}{1 + m\frac{2}{1}},$ while putting $y = m_2 x$ leads to a limit $\frac{m_2}{1 + m\frac{2}{2}}$. Similarly, letting $x \to 0$ while *y* remains constant or vice-versa leads to zero limit. Thus, we are led to erroneous results. Geometrically speaking when we approach the point

(0, 0) along different paths, first along lines with slopes m_1 and m_2 and then along lines parallel to the coordinate axes, the function reaches different limits. The simultaneous limit postulates that by whatever path the point is approached, the function *f* attains the same limit. In general the determination whether a simultaneous limit exists or not is a difficult matter but very often a simple consideration enables us to show that the limit does not exist.

It may however be noted that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \Rightarrow \lim_{x\to x_0} f(x,y_0) = L = \lim_{y\to y_0} f(x_0,y)$$

Non-existence of limit : The above remark makes it amply clear that if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and if $y = \phi(x)$ is any function such that $\phi(x) \to y_0$, when $x \to x_0$, then $\lim_{x\to x_0} f(x,\phi(x))$ must exist and should be equal to *L*. Thus, if we can find two functions $\phi_1(x)$ and $\phi_2(x)$ such that the limits of $f(x,\phi_1(x))$ and $f(x,\phi_2(x))$ are different, then the simultaneous limit in question does not exist.

Example 3 : Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, \text{ if } x^4 + y^2 \neq 0 \\ 0, \qquad \text{if } x^4 + y^2 = 0 \end{cases}$$

Solution: If we approach the origin along any axis, *f* (*x, y*) = 0. If approach (0,0) along any line *y* = *mx*, then

$$f(x,y) = f(x,mx) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \to 0, \text{ as } x \to 0.$$

So any straight line approach gives,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

But putting $y = mx^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx^2) = \frac{m}{1+m^2}$$

which is different for the different *m* selected.

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Thus, the function possesses no limit at the origin, but a straight line approach gives the limit zero.



CHECK YOUR PROGRESS 2

Q 1: Show, by using the definition that

$$\lim_{(x,y)\to(1,2)} 3xy = 6$$

Q 2: Show that

(a) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0,$ (b) $\lim_{(x,y)\to(0,0)} (x+y) = 0$

(c) $\lim_{(x,y)\to(0,0)} \frac{x\sin\left(x^2+y^2\right)}{x^2+y^2} = 0,$

(d) $\lim_{(x,y)\to(4,\pi)} x^2 \sin\left(\frac{y}{8}\right) = 8\sqrt{2}$

Continuity of function of several variables :

Definition : A function *f* is said to be continuous at a point (x_0, y_0) of its domain of definition, if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

In other words, a function *f* is said to be continuous at a point (x_0, y_0) of its domain of definition if for every $\in > 0,$ there exists a neighbourhood *N* of (x_0, y_0) such that

$$\left|f(x,y) - f(x_0,y_0)\right| < \in, \text{ for all } (x,y) \in N.$$

(11)

Example 4 : Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

Solution: Let $x = r\cos\theta$ and $y = r\sin\theta$

Then

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = r\,|\sin\theta\cos\theta| \leq r = \sqrt{x^2+y^2} < \in,$$

if

$$x^2 < \frac{\in^2}{2},\ y^2 < \frac{\in^2}{2}$$

or, if $|x|<\frac{\in}{\sqrt{2}}, |y|<\frac{\in}{\sqrt{2}}$

Thus $\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|<\in$, when $|x|<\frac{\in}{\sqrt{2}}, |y|<\frac{\in}{\sqrt{2}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Therefore, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = f(0,0)$

Hence, *f* is continuous at (0, 0).

Note :

1. The definition of continuity of a function *f* at a point (x_0, y_0) requires that besides (x_0, y_0), *f* is defined in a certain neighbourhood of (x_0, y_0) and moreover the limit of *f* when $(x, y) \to (x_0, y_0)$ exists and equals to the value of *f* (x_0, y_0).
2. A function which is not continuous at a point is said to be discontinuous there at.

Remarks: A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of a single variable. To be more specific if a function *f* of two variables *x, y* is continuous at (x_0, y_0), then $f(x, y_0)$ is a continuous function of *x* at $x = x_0$ and $f(x_0, y)$ that of *y* at $y = y_0$.

The converse however is not true, i.e., a function may be a continuous function of one variable when the others remain constant and yet not be a continuous function of all the variables.

For instance, consider a function *f* where

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y}, \text{ if } (x,y) \neq (0,0) \\ 0, \quad \text{if } (x,y) = (0,0) \end{cases}$$

The function is not continuous at (0, 0) for $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. But

$$\lim_{x\to 0} f(x,0) = 0 = f(0,0), \text{ and } \lim_{y\to 0} f(0,y) = 0 = f(0,0)$$

so that *f* is continuous at (0, 0) when considered as a function of a single variable *x* or that of *y*.

A function is said to be continuous in a region if it is continuous at every point of the same.

As in limits, it can be easily proved that the sum, difference, product and quotient (provided the denominator does not vanish) of two continuous functions are also continuous.

Example 5 : Investigate the continuity at (0, 0) of

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, \text{ if } (x,y)\neq(0,0)\\ 0, \quad \text{if } (x,y)=(0,0)\end{cases}$$

Solution: Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, therefore the function is not continuous at (0, 0).

Example 6 : Investigate for continuity at (1, 2) of

$$f(x,y)=\begin{cases}x^2+2y, \text{ if } (x,y)\neq(1,2)\\ 0, \quad \text{if } (x,y)=(1,2)\end{cases}$$

Solution: Here,

$$\lim_{(x,y)\to(1,2)} f(x,y)=\lim_{(x,y)\to(1,2)}(x^2+y)=5\neq f(1,2)$$

Hence, the function is not continuous at (1, 2).

The point (1, 2) is a point of discontinuity of the function.

However, if the function has the value 5 at (1, 2), it was then continuous at the point.



CHECK YOUR PROGRESS 3

Q 1: Show that the following functions are continuous at the origin:

(a) $$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, \text{ if } (x,y)\neq(0,0)\\ 0, \quad \text{if } (x,y)=(0,0)\end{cases}$$

(b) $$f(x,y)=\begin{cases}\dfrac{x^3y^3}{x^2+y^2}, \text{ if } (x,y)\neq(0,0)\\ 0, \quad \text{if } (x,y)=(0,0)\end{cases}$$

Q 2: Show that the following functions are discontinuous at the origin:

(a) $$f(x,y)=\begin{cases}\dfrac{1}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

(b) $$f(x,y)=\begin{cases}\dfrac{x^4-y^4}{x^4+y^4}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

(c) $$f(x,y)=\begin{cases}\dfrac{x^2y}{x^3+y^3}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

(d) $$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^4+y^4}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

(e) $$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & \text{if } x\neq y\\ 0, & \text{if } x=y\end{cases}$$

(f) $$f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^6}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

10.6 DIRECTIONAL DERIVATIVES

Definition : The derivative of the function *f* at $P_0(x_0,y_0)$ in the direction of the unit vector $u=u_1i+u_2j$ is the number

$$\left(\frac{df}{ds}\right)_{u.P_0}=\lim_{s\to 0}\frac{f(x_0+su_1,y_0+su_2)-f(x_0,y_0)}{s} \qquad (12)$$

provided the limit exists.

The directional derivative is also denoted by $(D_uf)_{P_0}$ and called "the derivative of *f* at P_0 in the direction of *u*".

Example 7 : Using the definition of directional derivative, find the derivative of

$$f(x,y)=x^2+xy$$

at P_0 (1, 2) in the direction of the unit vector $u = \left(\frac{1}{\sqrt{2}}\right) i + \left(\frac{1}{\sqrt{2}}\right) j$

Solution: $\left(\frac{df}{ds}\right)_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1,\, y_0 + su_2) - f(x_0, y_0)}{s}$

$$= \lim_{s \to 0} \frac{f\left(1 + \frac{s}{\sqrt{2}},\, 2 + \frac{s}{\sqrt{2}}\right) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right) + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1.2)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\left(\frac{5s}{\sqrt{2}} + s^2\right)}{s}$$

$$= \lim_{s \to 0} \left(\frac{5}{\sqrt{2}} + s\right) = \left(\frac{5}{\sqrt{2}} + 0\right) = \frac{5}{\sqrt{2}}$$

Therefore, the rate of change of $f(x, y) = x^2 + xy$ at P_0 (1, 2) in the direction of the unit vector $u = \left(\frac{1}{\sqrt{2}}\right) i + \left(\frac{1}{\sqrt{2}}\right) j$ is $\frac{5}{\sqrt{2}}$.

10.7 PARTIAL DERIVATIVES

Definition : The ordinary derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. Partial derivative of *f* (*x, y*) with respect to *x* is generally denoted by

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

and

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

when these limits exist.

The partial derivatives at a particular point (x_0, y_0) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(x_0, y_0)}, \frac{\partial f(x_0, y_0)}{\partial x} \text{ or } f_x(x_0, y_0)$$

and $$\left[\frac{\partial f}{\partial y}\right]_{(x_0, y_0)}, \frac{\partial f(x_0, y_0)}{\partial y} \text{ or } f_y(x_0, y_0)$$

Also, $$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0)}{k}$$

in case the limit exists.

Example 7 : If $f(x, y) = 2x^2 - xy + 2y^2$, then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (1, 2).

Solution: Now

$$\frac{\partial f}{\partial x} = 4x - y = 2, \text{ at (1, 2),}$$

$$\frac{\partial f}{\partial y} = -x + 4y = 7 \text{ at (1, 2).}$$

Higher partial derivatives : Notice that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are themselves functions of two variables, so they can also be partially differentiated.

For a function of two variables $f : D \subset \mathrm{R}^2 \to \mathrm{R}$ there are four possibility :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Higher order partial derivatives are defined similarly :

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = f_{xyx}$$

Usually, $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For the functions we will been countering the mixed partial derivatives will generally be equal. In fact, this is true whenever f_{yx} and f_{xy} are continuous.



CHECK YOUR PROGRESS 4

Q 1: Find all the second partial derivatives of

$$f(x,y) = x^2 e^{-2y} + y^{-2}\cos x.$$

Gradient Vector

Definition: The gradient vector (gradiant) of *f* (*x, y*) at a point P_0 (x_0, y_0) is the vector

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

obtained by evaluating the partial derivatives of *f* at P_0.

Theorem 4 : If *f* (*x, y*) is differentiable in an open region containing P_0 (x_0, y_0), then

$$\left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0}.u, \qquad (13)$$

the dot product of the gradient *f* at P_0 and *u*.

Proof : We begin with the line

$$x = x_0 + su_1, \quad y = y_0 + su_2 \qquad (14)$$

through P_0 (x_0, y_0), parametrized with the arc length parameter *s* increasing in the direction of the unit vector $u = u_1 i + u_2 j$. Then, by applying the chain rule for differentiable of *f*

$$\left(\frac{df}{ds}\right)_{u,P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2, \quad \text{(Applying Eq. (14))}$$

$$= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0} i + \left(\frac{\partial f}{\partial y}\right)_{P_0} j\right].\left[u_1 i + u_2 j\right]$$

$$= (\nabla f)_{P_0} . u \quad \text{(Proved)}$$

Example 7 : Using the gradient formula find the directional derivative of

$$f(x, y) = xe^y + \cos(xy)$$

at the point (2, 0) in the direction $v = 3i - 4j$

Solution: The direction of v is the unit vector obtained by dividing v by its length:

$$u = \frac{v}{|v|} = \frac{v}{5} = \frac{3}{5} i - \frac{4}{5} j$$

The partial derivatives of f are everywhere continuous and at (2, 0) are given by

$$f_x(2,0) = \left(e^y - y\sin(xy)\right)_{(2,0)} = \left(e^0 - 0\right) = 1$$

$$f_y(2,0) = \left(xe^y - x\sin(xy)\right)_{(2,0)} = \left(2e^0 - 2.0\right) = 2$$

The gradient of f at (2, 0) is

$$\nabla f|_{(2,0)} = f_x(2,0)i + f_y(2,0)j = i + 2j$$

The derivative of f at (2, 0) in the direction of u is therefore

$$(D_u f)|_{(2,0)} = \nabla f|_{(2,0)} . u$$

$$= (i + 2j).\left(\frac{3}{5} i - \frac{4}{5} j\right) = \frac{3}{5} - \frac{8}{5} = -1 .$$

Evaluating the dot product in the formula

$$D_u f = \nabla f . u = |\nabla f \| u| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between the vectors u and ∇f, reveals the following properties :

1. The function f increases most rapidly when $\cos\theta = 1$ or when u is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

   $$D_u f = |\nabla f| \cos(0) = |\nabla f|$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is

   $$D_u f = |\nabla f| \cos(\pi) = -|\nabla f|$$

3. Any direction u orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and

$$D_u f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = |\nabla f| .0 = 0$$



CHECK YOUR PROGRESS 5

Q 1: Find the derivative of the following function at P_0 in the direction of *v*:

(a) $f(x,y) = 2xy - 3y^2,\ P_0(5,5),\ v = 4i + 3j$

(b) $f(x,y) = 2x^2 + y^2,\ P_0(-1,1),\ v = 3i - 4j$

(c) $f(x,y) = \cos(xy) + e^{y/2},\ P_0(1,0),\ v = 1i + 2j$

(d) $g(x,y) = x - \frac{y^2}{x} + \sqrt{3}\sec^{-1}(2xy),\ P_0(1,1),\ v = 12i + 5j$

Q 2: Find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the following functions in these directions :

(a) $f(x,y) = x^2 + xy + y^2,\ P_0(-1,1)$

(b) $f(x,y) = x^2 y + e^{xy}\sin y,\ P_0(1,0)$

(c) $g(x,y) = y - x^2,\ P_0(-1,0)$

Q 3: If $f(x,y) = x^3 y + e^{xy^2}$, find f_x and f_y.

Q 4: If $f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}$, when $x^2 + y^2 \neq 0$, and *f* (0, 0) = 0, show that

$$f_x(x,0) = 0 = f_y(0,y)$$

$$f_x(0,y) = -y, f_y(x,0) = x$$



10.8 LET US SUM UP

- The set R^n is defined as the set containing n-tuples $x = (x_1, x_2, ..., x_n)$ of real numbers. That is, R^n is the Cartesian product of R given as :

  $R^n = R \times R \times \ ... \times R$ (*n* times).

- The limit of the function $f : D \subset R^n \to R$, with $n \in N$, at the point $\hat{P} \in R^n$ is the number $L \in R$, denoted as $\lim_{P \to \hat{P}} f(P) = L$ if and only if for

every $\in > 0$ there exists $\delta > 0$ such that

$$0 < |P - \hat{P}| < \delta \Rightarrow |f(P) - L| < \in$$

- A function *f* is said to be continuous at a point (x_0, y_0) of its domain of definition, if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

In other words, a function *f* is said to be continuous at a point (x_0, y_0) of its domain of definition if for every $\in > 0$, there exists a neighbourhood *N* of (x_0, y_0) such that

$$|f(x,y) - f(x_0,y_0)| < \in, \text{ for all } (x,y) \in N.$$

The derivative of the function *f* at P_0 (x_0, y_0) in the direction of the unit vector $u = u_1 i + u_2 j$ is the number

$$\left(\frac{df}{ds}\right)_{u.P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

- The ordinary derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. Partial derivative of *f* (*x, y*) with respect to *x* is generally denoted by $\frac{\partial f}{\partial x}$ or f_x or $f_x(x,y)$, while those with respect to *y* are denoted $\frac{\partial f}{\partial y}$ or f_y or $f_y(x,y)$.

10.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Check your progress 2 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Check your progress 3 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Check your progress 4 :

Ans to Q No 1:

$$\frac{\partial f}{\partial x} = 3x^2 e^{-2y} - y^{-2}\sin x,$$

$$\frac{\partial f}{\partial y} = -2x^3 e^{-2y} - 2y^{-3}\cos x,$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2 e^{-2y} - y^{-2}\sin x\right) = 6xe^{-2y} - y^{-2}\cos x,$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-2x^3 e^{-2y} - 2y^{-3}\cos x\right) = 4x^3 e^{-2y} + 6y^{-4}\cos x,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2x^3 e^{-2y} - 2y^{-3}\cos x\right) = -6x^2 e^{-2y} + 2y^{-3}\sin x,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 e^{-2y} - y^{-2}\sin x\right) = -6x^2 e^{-2y} + 2y^{-3}\sin x$$

Check your progress 5 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Ans to Q No 4: Try yourself.



10.10 FURTHER READING

1) Walter, R. Principles of Mathematical Analysis. 3rd edition, McGraw-Hill, 2017.

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4) Apostol, T. M. Mathematical Analysis. 2nd edition. Wesley Publishing Co. 2002.

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6) Royden, H.L. Real Analysis, Macmillan Pub. Co., Inc. 4th edition, New York, 1993.

7) Bartle, R. G., Sherbert, D. R. (2011). Introduction to real analysis. Hoboken, NJ: Wiley.

8) Narayan, S., Raisinghania, M. D. Elements of Real Analysis. S. Chand & Company Pvt. Ltd. 2015.



10.11 MODEL QUESTIONS

Q 1: Show that the limit, when $(x,y)\to(0,0)$ does not exist in each case

(a) $\lim\dfrac{2xy}{x^2+y^2}$, (b) $\lim\dfrac{xy^3}{x^2+y^6}$,

(c) $\lim\dfrac{x^2y^2}{x^2y^2+\left(x^2-y^2\right)^2}$, (d) $\lim\dfrac{x^3+y^3}{x-y}$

Q 2: Show that the limit, when $(x,y)\to(0,0)$ exist in each case

(a) $\lim\dfrac{xy}{\sqrt{x^2+y^2}}$ (b) $\lim\dfrac{x^3y^3}{x^2+y^2}$

(c) $\lim\dfrac{x^3-y^3}{x^2+y^2}$ (d) $\lim\dfrac{x^4+y^4}{x^2+y^2}$

Q 3: Discuss the following functions for continuity at (0, 0) :

(a) $f(x,y)=\begin{cases}\dfrac{x^2y}{x^3+y^3}, & \text{if } x^2+y^2\neq 0\\ 0, & \text{if } x+y=0\end{cases}$

(b) $$f(x,y)=\begin{cases}2xy\dfrac{x^2-y^2}{x^2+y^2}, \text{ if } (x,y)\neq 0\\ 0, \quad \text{if } (x,y)=0\end{cases}$$

Q 4: Calculate $f_x, f_y, f_x(0,0), f_y(0,0)$ for the following :

(a) $$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, \ x\neq 0, \ y\neq 0\\ 0, \quad x=0=y\end{cases}$$

(b) $$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, \text{ if } x^2+y^2\neq 0\\ 0, \quad \text{if} \quad x=0=y\end{cases}$$

Q 5: If $$f(x,y)=\begin{cases}\dfrac{x^2-xy}{x+y}, \text{ if } (x,y)\neq(0,0)\\ 0, \quad \text{if} \quad (x,y)=(0,0)\end{cases}$$

find $f_x(0,0)$ and $f_y(0,0)$.

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